Note on the generalized Hansen and Laplace coefficients

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November 10, 2004

Abstract. Recently, Breiter *et al.*(2004) reported the computation of Hansen coefficients $X_k^{\gamma,m}$ for non integer values of γ. In fact, the Hansen coefficients are closely related to the Laplace $b_s^{(m)}$, and generalized Laplace coefficients $b_{s,r}^{(m)}$ (Laskar and Robutel, 1995) that do not require s, r to be integers. In particular, the coefficients $X_0^{\gamma,m}$ have very simple expressions in terms of the usual Laplace coefficients $b_{\gamma+2}^{(m)}$, and all their properties derive easily from the known properties of the Laplace coefficients.

Keywords: Keplerian motion, Hansen coefficients, Laplace coefficients, analytical methods

1. Introduction

The aim of this note is to clarify some simple relations between the Hansen coefficients, and the Laplace's, and generalized Laplace coefficients. Once these relations are explicited, the results quoted by Breiter *et al.*(2004) become simple translations of known results on the Laplace coefficients.

2. Hansen and Laplace coefficients

The Hansen coefficients (Hansen, 1855) are defined as the Fourier coefficients $X_k^{\gamma,m}$ of the series

$$\left(\frac{r}{a}\right)^\gamma \mathbf{e}^{imv} = \sum_{k=-\infty}^{+\infty} X_k^{\gamma,m}\mathbf{e}^{ikM} \tag{1}$$

where v, M are the true and mean anomaly, r, a the radial distance and semi-major axis. The transformation $v \to -v$ transforms M in $-M$. Thus $X_k^{\gamma,m}$ is real and $X_{-k}^{\gamma,-m} = X_k^{\gamma,m}$. We have

$$X_k^{\gamma,m} = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a}\right)^\gamma \mathbf{e}^{imv}\mathbf{e}^{-ikM}\,dM\ . \tag{2}$$

In particular, for $k = 0$,

$$X_0^{\gamma,m} = \frac{1}{\sqrt{1-e^2}}\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a}\right)^{\gamma+2}\mathbf{e}^{imv}\,dv\ . \tag{3}$$

Hansen (1855) uses the expressions in term of the true anomaly v

$$\frac{r}{a} = \frac{1-e^2}{1+e\cos v} = \frac{(1-e^2)(1+\beta^2)}{(1+\beta\xi)(1+\beta\xi^{-1})} \tag{4}$$

where $\xi = \mathbf{e}^{iv}$, and $\beta = (1-\sqrt{1-e^2})/e$. We thus obtain immediately the expansion of $(r/a)^{\gamma+2}$ in Laurent series of ξ

$$\left(\frac{r}{a}\right)^{\gamma+2} = (1-e^2)^{\gamma+2}(1+\beta^2)^{\gamma+2}\frac{1}{2}\sum_{k=-\infty}^{+\infty} b_{\gamma+2}^{(k)}(-\beta)\xi^k \tag{5}$$

where $b_s^k(\alpha)$ are the classical Laplace coefficients defined as the coefficients of the Laurent series

$$(1-\alpha z)^{-s}(1-\alpha z^{-1})^{-s} = \frac{1}{2}\sum_{k=-\infty}^{+\infty} b_s^{(k)}(\alpha) z^k \ , \tag{6}$$

with $b_s^{(k)}(-\alpha) = (-1)^k b_s^{(k)}(\alpha)$; $b_s^{(-k)}(\alpha) = b_s^{(k)}(\alpha)$, and for $k \geq 0$,

$$b_s^{(k)}(\alpha) = \frac{(s)_k}{k!}\alpha^k F(s, s+k, k+1; \alpha^2) \ , \tag{7}$$

where $(s)_0 = 1$, $(s)_k = s(s+1)\cdots(s+k-1)$ for $k \geq 1$. Thus

$$X_0^{\gamma,m} = \frac{(-1)^m}{2}(1-e^2)^{\gamma+3/2}(1+\beta^2)^{\gamma+2} b_{\gamma+2}^{(m)}(\beta) \ . \tag{8}$$

Any property of the Laplace coefficients can thus be translated into a property on the Hansen coefficients $X_0^{\gamma,m}$. In particular, in 1785, Laplace demonstrated the most useful relations

$$\begin{aligned} b_{s+1}^{(j)}(\alpha) \quad &= \frac{(s+j)}{s}\frac{(1+\alpha^2)}{(1-\alpha^2)^2} b_s^{(j)}(\alpha) - \frac{2(j-s+1)}{s}\frac{\alpha}{(1-\alpha^2)^2} b_s^{(j+1)}(\alpha) \\ b_{s+1}^{(j+1)}(\alpha) &= \frac{j}{j-s}(\alpha+\frac{1}{\alpha}) b_{s+1}^{(j)}(\alpha) - \frac{j+s}{j-s} b_{s+1}^{(j-1)}(\alpha) \end{aligned} \tag{9}$$

that are immediately translated as[1]

$$X_0^{\gamma,m} = \frac{\gamma+1+m}{\gamma+1} X_0^{\gamma-1,m} + \frac{m-\gamma}{\gamma+1} e X_0^{\gamma-1,m+1} \tag{10}$$

$$X_0^{\gamma,m+1} = -\frac{2}{e}\frac{m}{m-\gamma-1} X_0^{\gamma,m} - \frac{m+\gamma+1}{m-\gamma-1} X_0^{\gamma,m-1} \ . \tag{11}$$

As with the Laplace coefficients, these relations allow to express all coefficients with respect to the two first ones $X_0^{\gamma,0}$ and $X_0^{\gamma,1}$. Breiter *et al.*(2004) treat as a special case $\gamma = (2n+1)/2$. This is precisely the case of the expansion of the Newtonian potential in Laplace coefficients. In fact, the recurrence formulas of Laplace (10, 11), allow to express all coefficients not with 4 initial coefficients, as quoted in Breiter *et al.*, 2004, but from only two of them, namely

$$X_0^{-3/2,0} = (1+\beta^2)^{1/2}\frac{1}{2} b_{1/2}^{(0)}(\beta) \ ; \qquad X_0^{-3/2,1} = -(1+\beta^2)^{1/2}\frac{1}{2} b_{1/2}^{(1)}(\beta) \ , \tag{12}$$

[1] Eq. 11 is the same as Eq. (19) of (Breiter *et al.*, 2004).

with the expressions in terms of the elliptic integrals of first and second kind $K(\beta), E(\beta)$ (Tisserand, 1889)

$$b_{1/2}^{(0)}(\beta) = \frac{4}{\pi} K(\beta) \ ; \qquad b_{1/2}^{(1)}(\beta) = \frac{4}{\pi\beta}(K(\beta) - E(\beta)) \ . \tag{13}$$

More generally, $X_0^{\gamma,m}$ can be expressed with respect to $X_0^{\gamma-[\gamma],0}$ and $X_0^{\gamma-[\gamma],1}$, where $[\gamma]$ denotes the integer part of γ.

3. Hansen coefficients for $k \in \mathbb{Z}$

The expressions of the Hansen coefficients for $k \neq 0$ are obtained in a similar way using expansions of (2) in eccentric anomaly E as

$$X_k^{\gamma,m} = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{r}{a}\right)^{\gamma+1} \mathbf{e}^{imv} \mathbf{e}^{ike\sin E} \mathbf{e}^{-ikE} dE \ . \tag{14}$$

With $\eta = \mathbf{e}^{iE}$ and

$$\frac{r}{a} = \frac{1}{1+\beta^2}(1-\beta\eta)(1-\beta\eta^{-1}); \quad \mathbf{e}^{iv} = \eta \frac{(1-\beta\eta^{-1})}{(1-\beta\eta)} \ , \tag{15}$$

we have

$$\left(\frac{r}{a}\right)^{\gamma+1} \mathbf{e}^{imv} = \frac{\eta^m}{2(1+\beta^2)^{\gamma+1}} \sum_{l=-\infty}^{+\infty} b_{-\gamma-1+m,-\gamma-1-m}^{(l)}(\beta)\eta^l \tag{16}$$

where $b_{s,r}^{(k)}$ are the generalized Laplace coefficients defined in (Laskar and Robutel, 1995) as the coefficients of the Laurent series

$$(1-\alpha z)^{-s}(1-\alpha z^{-1})^{-r} = \frac{1}{2} \sum_{k=-\infty}^{+\infty} b_{s,r}^{(k)}(\alpha) z^k \ . \tag{17}$$

We have $b_{s,r}^{(k)}(-\alpha) = (-1)^k b_{s,r}^{(k)}(\alpha)$; $b_{s,r}^{(-k)}(\alpha) = b_{r,s}^{(k)}(\alpha)$, and for $k \geq 0$

$$b_{s,r}^{(k)}(\alpha) = \frac{(s)_k}{k!} \alpha^k F(r, s+k, k+1; \alpha^2) \ . \tag{18}$$

The classical expansion in Bessel functions

$$\mathbf{e}^{ike\sin E} = \sum_{n=-\infty}^{+\infty} J_n(ke)\eta^n \ , \tag{19}$$

allows the computation of the Hansen coefficients $X_k^{\gamma,m}$ in terms of Bessel functions and Laplace coefficients as

$$X_k^{\gamma,m} = \frac{1}{2(1+\beta^2)^{\gamma+1}} \sum_{n=-\infty}^{+\infty} b_{-\gamma-1+m,-\gamma-1-m}^{(k-n-m)}(\beta) J_n(ke) \ . \tag{20}$$

4. Recurrence relations

As for the Laplace coefficients, one can derive recurrence relations for the generalized Laplace coefficients. Multiplying Eq. (17) by $A = 1 - \alpha z$ and respectively $B = 1 - \alpha z^{-1}$, one derives the two recurrence relations

$$b^{(k)}_{s,r+1}(\alpha) - \alpha b^{(k+1)}_{s,r+1}(\alpha) = b^{(k)}_{s,r}(\alpha) \ ; \tag{21}$$

$$b^{(k)}_{s+1,r}(\alpha) - \alpha b^{(k-1)}_{s+1,r}(\alpha) = b^{(k)}_{s,r}(\alpha) \ . \tag{22}$$

The derivative of (17) with respect to z gives also

$$\alpha s A^{-s-1}B^{-r} - \alpha r z^{-2} A^{-s} B^{-r-1} = \frac{1}{2}\sum_{k=-\infty}^{+\infty} k\, b^{(k)}_{s,r}(\alpha) z^{k-1} \ . \tag{23}$$

This relation provides directly the recurrence relation

$$\alpha s\, b^{(k-1)}_{s+1,r}(\alpha) - \alpha r\, b^{(k+1)}_{s,r+1}(\alpha) = k b^{(k)}_{s,r}(\alpha) \ , \tag{24}$$

but if we express $A^{-s-1}B^{-r}$ in terms of $b^{(k)}_{s,r}(\alpha)$ and $A^{-s}B^{-r-1}$ in terms of $b^{(k)}_{s,r+1}(\alpha)$, one obtains with (21) the relation

$$(1-\alpha^2) r b^{(k)}_{s,r+1}(\alpha) = (r-k) b^{(k)}_{s,r}(\alpha) + \alpha(s+k-1) b^{(k-1)}_{s,r}(\alpha) \ . \tag{25}$$

Conversely, when $A^{-s-1}B^{-r}$ is expressed in terms of $b^{(k)}_{s+1,r}(\alpha)$ and $A^{-s}B^{-r-1}$ in term of $b^{(k)}_{s,r}(\alpha)$, with (22), we have

$$(1-\alpha^2) s b^{(k)}_{s+1,r}(\alpha) = (s+k) b^{(k)}_{s,r}(\alpha) + \alpha(r-k-1) b^{(k+1)}_{s,r}(\alpha) \ . \tag{26}$$

Finally, when $A^{-s-1}B^{-r}$ and $A^{-s}B^{-r-1}$ are expressed in terms of $b^{(k)}_{s,r}(\alpha)$, we have

$$(r-k-1) b^{(k+1)}_{s,r}(\alpha) = (s+k-1) b^{(k-1)}_{s,r}(\alpha) + (\alpha(r-s-k) - \frac{k}{\alpha}) b^{(k)}_{s,r}(\alpha) \ . \tag{27}$$

The five relations (21, 22, 25, 26, 27) allow then to express any generalized Laplace coefficient $b^{(k)}_{s,r}(\alpha)$ in terms of only two of them, namely

$$b^{(0)}_{s-[s],r-[r]}(\alpha) \qquad \text{and} \qquad b^{(1)}_{s-[s],r-[r]}(\alpha) \ . \tag{28}$$

In particular, all generalized Laplace coefficients involved in Eq. (20) can be expressed in terms of the two Laplace coefficients $b^{(0)}_{[\gamma]-\gamma}(\alpha)$ and $b^{(1)}_{[\gamma]-\gamma}(\alpha)$.

5. Expressions in terms of true and eccentric anomaly

The general Hansen coefficients $X_k^{\gamma,m}$ can be expressed in simple form with respect to the Bessel functions $J_n(ke)$ and the generalized Laplace coefficients $b_{s,r}^{(k)}(\beta)$ (20), but although recurrence relations exist for both functions, they will not translate easily into recurrence relations for the Hansen coefficients $X_k^{\gamma,m}$ when $k \neq 0$. On the opposite, in the expression in terms of the true anomaly

$$\left(\frac{r}{a}\right)^{\gamma} \mathbf{e}^{imv} = \sum_{k=-\infty}^{+\infty} Y_k^{\gamma,m} \mathbf{e}^{ikv} \ , \tag{29}$$

or of the eccentric anomaly

$$\left(\frac{r}{a}\right)^{\gamma} \mathbf{e}^{imv} = \sum_{k=-\infty}^{+\infty} Z_k^{\gamma,m} \mathbf{e}^{ikE} \ , \tag{30}$$

the Fourier coefficients $Y_k^{\gamma,m}$ and $Z_k^{\gamma,m}$ (see also Brumberg, 1995) have very simple form in terms of Laplace and generalized Laplace coefficients. Indeed, from (5) and (16), we have

$$Y_k^{\gamma,m} = \frac{(-1)^{k-m}}{2} (1-e^2)^{\gamma} (1+\beta^2)^{\gamma} b_{\gamma}^{(k-m)}(\beta) \ ; \tag{31}$$

$$Z_k^{\gamma,m} = \frac{1}{2(1+\beta^2)^{\gamma}} b_{-\gamma+m,-\gamma-m}^{(k-m)}(\beta) \ . \tag{32}$$

All the previous recurrence relations on Laplace and generalized Laplace coefficients can then be translated into recurrence relations on the $(Y_k^{\gamma,m})$ and $(Z_k^{\gamma,m})$ coefficients.

6. Conclusion

The computation of Hansen (1855), or the later computation of Hill (1875) and Tisserand (1889), are very similar to the present presentation, the novelty here being the use of Laplace, and generalized Laplace coefficients to express the Hansen coefficients in a simple form. This explicits the fact that nowhere in the original demonstration of Hansen is requested the fact that γ is an integer. This is particularly visible for the computation of $X_0^{\gamma,m}$ that are very simple expressions of the Laplace coefficients $b_{\gamma+2}^{(m)}$. As for the usual Laplace coefficients, recurrence relations allow to express the generalized Laplace coefficients $b_{s,r}^{(k)}$ in terms of only two of them, $b_{s-[s],r-[r]}^{(0)}$ and $b_{s-[s],r-[r]}^{(1)}$, for example. The Laplace and generalized Laplace coefficients are also introduced naturally in the expression of $(r/a)^{\gamma} \exp(imv)$ in Fourier series of the true and eccentric anomaly (29–32).

Acknowledgements

The author is grateful to the referee for his suggestions and to J. Couetdic for a careful reading of the manuscript.

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